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An Examination on Sets of preopen topology

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Abstract

Utilizing the idea of pre-open set, we present and study topological properties of pre-limit points, pre-derived sets, pre-interior and pre-closure of a set, pre-inside points, pre-line, pre-frontier and pre-outside. The relations between pre-derived set (resp. pre-limit point, pre-inside (point), pre-line, pre-wilderness, and pre-exterior) and α-derived set (resp. α-limit point, α-inside (point), α-line, α-outskirts, and α-outside) are explored.

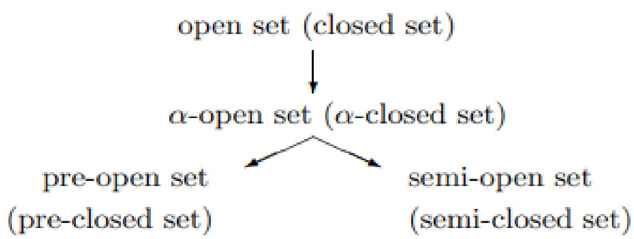
Keywords: Prelimit Point, Pre-derived sets, Pre- Closure, Pre-open sets, Topology

Introduction

The thought of α-open set was presented by Nastad. From that point forward it has been broadly explored in a few written works. In Caldas presented and concentrated on topological properties of α-derived, αborder, α-wilderness, and α-outside of a set by utilizing the idea of α-open sets. The idea of pre-open set was presented by Mashhour et al. In this paper, we present the ideas of pre-limit points, pre-derived sets, pre-inside and pre-closure of a set, pre-inside points, pre-line, pre-wilderness and pre-outside by utilizing the idea of pre-open sets, and study their topological properties. We give relations between pre-derived set (resp. pre-limit point, pre-inside (point), pre-line, pre-boondocks, and pre-outside) and α-derived set (resp. αlimit point, α-inside (point), α-line, α-boondocks, and α-outside)

2. Preliminaries

Through this paper, $(X, \mathscr{T})$ and $(Y, \mathscr{K})$ (simply X and Y) always mean topological spaces. A subset A of X is said to be *pre-open* [11] (respectively, *α-open* [14] and *semi-open* [13]) if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ (respectively, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and $A \subset \mathrm{Cl}(\mathrm{Int}(A))$). The complement of a pre-open set (respectively, an α-open set and a semi-open set) is called a *pre-closed set* (respectively, an *α-closed set* and a *semi-closed set*). The intersection of all pre-closed sets (respectively, α-closed sets and semi-closed sets) containing A is called the *pre-closure* (respectively, *α-closure* and *semi-closure*) of A, denoted by $\mathrm{Cl}_p(A)$ (respectively, $\mathrm{Cl}_\alpha(A)$ and $\mathrm{Cl}_s(A)$). A subset A is also pre-closed (respectively, α-closed and semi-closed) if and only if $A = \mathrm{Cl}_p(A)$ (respectively, $A = \mathrm{Cl}_\alpha(A)$ and $A = \mathrm{Cl}_s(A)$). We denote the family of pre-open sets (respectively, α-open sets and semi-open sets) of $(X, \mathscr{T})$ by $\mathscr{T}^p$ (respectively, $\mathscr{T}^\alpha$ and $\mathscr{T}^s$). Obviously, we have the following relations.



None of these implications is reversible in general.

3. Pre-open sets and α-open sets

Definition 3.1 ([11, 14]). A subset A of X is said to be *pre-open* (respectively, *α-open*) if $A \subseteq \mathrm{Int}(\mathrm{Cl}A)$ (respectively, $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}A))$).

The complement of a pre-open set (respectively, an α-open set) is called a *pre-closed set* (respectively, an *α-closed set*).

The intersection of all pre-closed sets (respectively, α-closed sets) containing A is called the *pre-closure* (respectively, *α-closure*) of A, denoted by $\mathrm{Cl}_p(A)$ (respectively, $\mathrm{Cl}_\alpha(A)$).

A subset A is also pre-closed (respectively, α-closed) if and only if $A = \mathrm{Cl}_p(A)$ (respectively, $A = \mathrm{Cl}_\alpha(A)$). We denote the family of pre-open sets (respectively, α-open sets) of $(X, \mathscr{T})$ by $\mathscr{T}^p$ (respectively, $\mathscr{T}^\alpha$).

Example 3.2. Let $\mathscr{T} = \{\varnothing, X, \{a\}, \{c,d\}, \{a,c,d\}\}$ be a topology on $X = \{a,b,c,d,e\}$. Then we have

$$\mathscr{T}^\alpha = \mathscr{T} \cup \{\{a,b,c,d\}, \{a,c,d,e\}\},$$

$$\mathscr{T}^p = \mathscr{T} \cup \{\{c\}, \{d\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,e\}, \{a,d,e\}, \{a,b,c,d\}, \{a,b,c,e\}, \{a,b,d,e\}, \{a,c,d,e\}\}.$$

4. Applications of pre-open sets

Definition 4.1. Let A be a subset of a topological space $(X, \mathscr{T})$. A point $x \in X$ is said to be *pre-limit point* (resp. *α-limit point*) of A if it satisfies the following assertion:

$$(\forall G \in \mathscr{T}^p(\text{ resp. } \mathscr{T}^\alpha))\,(x \in G \Rightarrow G \cap (A \setminus \{x\}) \neq \varnothing).$$

The set of all pre-limit points (resp. α-limit points) of A is called the *pre-derived set* (resp. *α-derived set*) of A and is denoted by $D_p(A)$ (resp. $D_\alpha(A)$). Denote by $D(A)$ the derived set of A.

Note that for a subset A of X, a point $x \in X$ is not a pre-limit point of A if and only if there exists a pre-open set G in X such that

$$x \in G \quad \text{and} \quad G \cap (A \setminus \{x\}) = \varnothing$$

or, equivalently,

$$x \in G \quad \text{and} \quad G \cap A = \varnothing \quad \text{or} \quad G \cap A = \{x\}$$

or, equivalently,

$$x \in G \quad \text{and} \quad G \cap A \subseteq \{x\}.$$

Example 4.2. Let $X = \{a, b, c\}$ with topology $\mathscr{T} = \{X, \varnothing, \{a\}\}$. Then we have the followings:

(i) $\mathscr{T}^p = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}\} = \mathscr{T}^\alpha$.
(ii) If $A = \{c\}$, then $D(A) = \{b\}$ and $D_\alpha(A) = D_p(A) = \varnothing$.
(iii) If $B = \{a\}$ and $C = \{b, c\}$, then $D_p(B) = \{b, c\}$, $D_p(C) = \varnothing$ and $D_p(B \cup C) = \{b, c\}$.

Theorem 4.3. *If a topology $\mathscr{T}$ on a set X contains only $\varnothing$, X, and $\{a\}$ for a fixed $a \in X$, then $\mathscr{T}^p = \mathscr{T}^\alpha$.*

Proof. Let $a \in X$ and let A be an element of $\mathscr{T}^p$. Then $a \in A$. In fact, if not then $A \nsubseteq \mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Int}(\{a\}^c) = \varnothing$. Hence $A \notin \mathscr{T}^p$, a contradiction. Now since $\mathrm{Int}(A) = \{a\}$, we have

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) = \mathrm{Int}(\mathrm{Cl}(\{a\})) = \mathrm{Int}(X) = X$$

which contains A, that is, $A \in \mathscr{T}^\alpha$. Note that $\mathscr{T}^\alpha \subseteq \mathscr{T}^p$. Thus $\mathscr{T}^\alpha = \mathscr{T}^p$. $\square$

Example 4.4. Let $X = \{a, b, c, d, e\}$ with topology

$$\mathscr{T} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Then

$$\begin{aligned} \mathscr{T}^p = \; & \{X, \varnothing, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \\ & \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \\ & \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \\ & \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\} \end{aligned}$$

$$\begin{aligned}\mathscr{T}^\alpha &= \{X, \varnothing, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{c,d,e\},\\ &\quad \{a,b,c,d\}, \{a,c,d,e\}, \{b,c,d,e\}\}.\end{aligned}$$

Consider subsets $A = \{a, b, c\}$ and $B = \{b, d\}$ of X. Then

$$\begin{array}{ll} D(A) = \{b,d,e\}, & D_p(A) = \varnothing, \\ \operatorname{Int}(A) = \{a\}, & \operatorname{Int}_p(A) = A, \\ \operatorname{Int}_\alpha(A) = \{a\}, & \operatorname{Cl}_p(A) = A, \\ \operatorname{Cl}_\alpha(A) = X, & \operatorname{Cl}_p(B) = B, \\ \operatorname{Cl}_\alpha(B) = \{b,c,d,e\}, & \operatorname{Int}(B) = \varnothing, \\ \operatorname{Int}_p(B) = B, & \operatorname{Int}_\alpha(B) = \varnothing. \end{array}$$

Example 4.5. Consider a topology

$$\mathscr{T} = \{X, \varnothing, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$$

on $X = \{a, b, c, d, e\}$. Then

$$\begin{aligned}\mathscr{T}^p &= \{X, \varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,b,c\},\\ &\quad \{a,b,d\}, \{a,b,e\}, \{a,c,d\}, \{a,c,e\}, \{a,d,e\}\\ &\quad \{a,b,c,d\}, \{a,b,c,e\}, \{a,b,d,e\}, \{a,c,d,e\}\}\\ &= \mathscr{T}^\alpha.\end{aligned}$$

For subsets $A = \{c, d, e\}$ and $B = \{b\}$ of X, we have

$$\begin{array}{ll} D(A) = \{c,d\} & D(B) = \{e\}. \\ D_p(A) = \varnothing & D_p(B) = \varnothing. \\ D_\alpha(A) = \varnothing & D_\alpha(B) = \varnothing. \\ \operatorname{Int}(A) = \varnothing & \operatorname{Int}(B) = \varnothing, \\ \operatorname{Int}_p(A) = \varnothing, & \operatorname{Int}_p(B) = \varnothing, \\ \operatorname{Int}_\alpha(A) = \varnothing, & \operatorname{Int}_\alpha(B) = \varnothing, \\ \operatorname{Cl}_p(A) = \{c,d,e\}, & \operatorname{Cl}_p(B) = \{b\}, \\ \operatorname{Cl}_\alpha(A) = \{c,d,e\}, & \operatorname{Cl}_\alpha(B) = \{b\}, \\ \operatorname{Cl}_p(\{b,d\}) = \{b,d\}, & \operatorname{Cl}_\alpha(\{b,d\}) = \{b,d\}, \\ \operatorname{Int}(\{b,d\}) = \varnothing, & \operatorname{Int}_p(\{b,d\}) = \varnothing, \\ \operatorname{Int}_\alpha(\{b,d\}) = \varnothing. & \end{array}$$

Lemma 4.6. *If there exists $a \in X$ such that $\{a\}$ is the smallest element of $(\mathscr{T} \setminus \{\varnothing\}, \subseteq)$, then every non-empty pre-open set contains $\bigcap\{G_i \mid G_i \in \mathscr{T} \setminus \{\varnothing\}; i = 1, 2, 3, \cdots\}$.*

Proof. If $\{a\}$ is the smallest element of $(\mathscr{T} \setminus \{\varnothing\}, \subseteq)$, then

$$\bigcap\{G_i \mid G_i \in \mathscr{T} \setminus \{\varnothing\}; i = 1, 2, 3, \cdots\} = \{a\}.$$

Let A be a non-empty pre-open set in X. If $a \notin A$, then $\operatorname{Cl}(A) \subseteq \{a\}$ and so

$$A \nsubseteq \operatorname{Int}(\operatorname{Cl}(A)) \subseteq \operatorname{Int}(\{a\}^c) = \varnothing$$

which is a contradiction. Hence $a \in A$, and so the desired result is valid. □

Theorem 4.7. *Let $\mathscr{T}$ be a topology on a set X. If there exists $a \in X$ such that $\{a\}$ is the smallest element of $(\mathscr{T} \setminus \{\varnothing\}, \subseteq)$, then $\mathscr{T}^{\alpha} = \mathscr{T}^{p}$.*

Proof. It is sufficient to show that $\mathscr{T}^{p} \subseteq \mathscr{T}^{\alpha}$. Let $A \in \mathscr{T}^{p}$. If $A = \varnothing$, then clearly $A \in \mathscr{T}^{\alpha}$. Assume that $A \neq \varnothing$. Then $a \in A$ by Lemma 4.6. Since $\{a\} \subseteq \mathrm{Int}(A)$, it follows that $X = \mathrm{Cl}(\{a\}) \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ so that

$$A \subseteq X = \mathrm{Int}(X) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))).$$

Hence A is an α-open set. □

Theorem 4.8. *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies on X such that $\mathscr{T}_1^{p} \subseteq \mathscr{T}_2^{p}$. For any subset A of X, every pre-limit point of A with respect to $\mathscr{T}_2$ is a pre-limit point of A with respect to $\mathscr{T}_1$.*

Proof. Let x be a pre-limit point of A with respect to $\mathscr{T}_2$. Then $(G \cap A) \setminus \{x\} \neq \varnothing$ for every $G \in \mathscr{T}_2^{p}$ such that $x \in G$. But $\mathscr{T}_1^{p} \subseteq \mathscr{T}_2^{p}$, so, in particular, $(G \cap A) \setminus \{x\} \neq \varnothing$ for every $G \in \mathscr{T}_1^{p}$ such that $x \in G$. Hence x is a pre-limit point of A with respect to $\mathscr{T}_1$. □

The converse of Theorem 4.8 is not true in general as seen in the following example.

Example 4.9. Consider topologies $\mathscr{T}_1 = \{X, \varnothing, \{a\}\}$ and

$$\mathscr{T}_2 = \{X, \varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$$

on a set $X = \{a, b, c, d\}$. Then

$$\mathscr{T}_1^{p} = \mathscr{T}_1 \cup \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$$

and

$$\mathscr{T}_2^{p} = \mathscr{T}_2 \cup \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}.$$

Note that $\mathscr{T}_1^{p} \subseteq \mathscr{T}_2^{p}$ and c is a pre-limit point of $A = \{a, b\}$ with respect to $\mathscr{T}_1$, but it is not a pre-limit point of A with respect to $\mathscr{T}_2$.

Conclusion

Let $(X, \mathscr{T})$ be a topological space which is given in Example 4.4. Take $A = \{d, e\}$. Then $\mathrm{Ext}_{\alpha}(A) = \{a\}$ and $\mathrm{Ext}_{p}(A) = \{a, b, c\}$. Thus the reverse inclusion of Theorem 4.45(1) is not valid. Let $A = \{b, e\}$ and $B = \{c, d, e\}$. Then $\mathrm{Ext}_{p}(B) = \{a\} \subseteq \{a, c, d\} = \mathrm{Ext}_{p}(A)$. This shows that the converse of (5) in Theorem 4.45 is not valid. Now let $A = \{d, e\}$ and $B = \{c\}$. Then $\mathrm{Ext}_{p}(A \cup B) = \{a\} \neq \{a, b\} = \{a, b, c\} \cap \{a, b, d, e\} = \mathrm{Ext}_{p}(A) \cap \mathrm{Ext}_{p}(B)$ which shows that the equality in Theorem 4.45(6) is not valid. Finally let $A = \{a, b\}$ and $B = \{c, d, e\}$. Then $\mathrm{Ext}_{p}(A \cap B) = \{a, b, c, d, e\}$ and $\mathrm{Ext}_{p}(A) \cup \mathrm{Ext}_{p}(B) = \{a, c, d, e\}$. This shows that the equality in Theorem 4.45(7) is not valid.

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